$\mathbb{X}$-stability conditions on Calabi-Yau-$\mathbb{X}$ categories

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Joint work with Akishi Ikeda

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Outline

1. Motivations
   - Mirror symmetry
   - Saito-Frobenius structure
   - Cluster theory
2. $\mathbb{X}$-stability conditions
   - Stability conditions
   - $\mathbb{X}$-stability conditions
   - N-reduction
   - q-stability conditions
3. Surface case
   - Quadratic differentials
   - Further studies

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Mirror symmetry

Kontsevich's homological MS:

$$\mathcal{D}^b \operatorname{Fuk}(X) \cong \mathcal{D}^b(\operatorname{Coh} X^\vee),$$

Geometric expectation:

$$\operatorname{Stab}^\circ \mathcal{D} \sim \mathcal{M}_{\mathrm{cpx}}(X)$$

Idea:

$$Z(S^\vee) = \int_S \Omega$$

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Stability condition via quadratic differentials

Let $\mathbf{S}$ be a Fomin-Shapiro-Thurston marked surface.

Theorem (Bridgeland-Smith)

$$\operatorname{Stab}^\circ \mathcal{D}_3(\mathbf{S}) / \operatorname{Aut} \cong \operatorname{Quad}_3(\mathbf{S}).$$

By understanding Aut, there is a upgraded version:

Theorem (King-Qiu)

Suppose $\mathbf{S}$ is unpunctured and $\mathbf{S}_\triangle$ be its decorated version. Then

$$\operatorname{Stab}^\circ \mathcal{D}_3(\mathbf{S}) \cong \operatorname{FQuad}_3^\circ(\mathbf{S}_\triangle)$$

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Let $\mathfrak{S}$ be a flat surface and $\mathcal{D}_\infty(\mathfrak{S}) := \mathrm{TFuk}(\mathfrak{S})$ its the topological Fuakaya category.

Theorem (Haiden-Katzarkov-Kontsevich)

$$\mathrm{Stab}^\circ \mathcal{D}_\infty(\mathfrak{S}) \cong \mathrm{FQuad}_\infty(\mathfrak{S}).$$

Remark

- *$\mathcal{D}_3(\mathbf{S})$ is Calabi-Yau-3, which can be embeded into a derived Fukaya category (Smith).*
- *$\mathcal{D}_\infty(\mathfrak{S})$ is not Calabi-Yau.*

Aim: Relating BS and HKK via q-deformation.

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Frobenius structure

Let Q be a Dynkin quiver.

Let $\mathfrak{h}$ be the Cartan subalgebra of the f.d. complex simple Lie algebra $\mathfrak{g}$ corresponding to Q and $\mathfrak{h}_{\mathrm{reg}}$ its regular part. Consider the 'Calabi-Yau-∞' category

$$\mathcal{D}_\infty(Q) := D^b(\mathbf{k}Q).$$

We expect

$$\operatorname{Stab}\mathcal{D}_\infty(Q) \cong \mathfrak{h}/W \tag{1}$$

for the Frobenius manifold $\mathfrak{h}/W$, where W is the Wely group. This has been proved by HKK for A_n (and BQS for A_2).

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Almost Frobenius structure

On the other hand, consider the Calabi-Yau-N category

$$\mathcal{D}_N(Q) := \mathcal{D}_{fd}(\Gamma_N Q).$$

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$$\operatorname{Stab} \mathcal{D}_N(Q) / \operatorname{ST}_N(Q) \cong \mathfrak{h}_{\mathrm{reg}}/W, \tag{2}$$

where $\operatorname{ST}_N(Q)$ is the spherical twist group, that can be identified with Artin/braid group Br_Q by Qiu-Woolf.
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Twisted periods

Upgraded version:

$$\begin{array}{ccc} \widetilde{\mathfrak{h}_{\mathrm{reg}}/W} & \xrightarrow{\ \phi_s\ } & \operatorname{Stab}^\circ \mathcal{D}_N(Q) \\ & {\scriptstyle P_\nu} \searrow \quad \swarrow {\scriptstyle \mathcal{Z}_N} & \\ & \operatorname{Hom}(\Gamma, \mathbb{C}) & \end{array}$$

where the period map P_ν corresponds to the central charge Z_N for

$$\nu = \frac{N-2}{2}.$$

Aim: generalize to $s = N$ in $\mathbb{C}$.

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Cluster categories

$N \geq 2$, τ be the Auslander-Reiten functor.

Definition (Buan-Marsh-Reineke-Reiten-Todorov, Keller)

For any integer $m \geq 2$, the m-cluster shift is $\Sigma_m = \tau^{-1} \circ [m-1]$. The m-cluster category $\mathcal{C}_m(Q)$ is the orbit category

$$\mathcal{C}_m(Q) := \mathcal{D}_\infty(Q)/\Sigma_m\,.$$

Theorem (Amoit-Guo-Keller)

Let $\Gamma_N Q$ be the Ginzburg dga of degree N and $\mathcal{C}(\Gamma_N Q) = \operatorname{per}\Gamma_N Q/\mathcal{D}_N(Q)$. Then

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Cluster-∞ categories

Naively:

$$\mathcal{D}_\infty(Q) = \lim_{m \to \infty} \mathcal{C}_m(Q).$$

The corresponding statement for the spaces of stability conditions is (cf. Qiu)

$$\operatorname{Stab} \mathcal{D}_\infty(Q) \cong \lim_{N \to \infty} \operatorname{Stab} \mathcal{D}_N(Q) / \operatorname{Br}_Q .$$

Application: $\mathcal{D}_\infty(Q)$ is a cluster-$\mathbb{X}$ category (and hence sliting is cluster-$\mathbb{X}$ tilting).

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$\mathbb{Z}$- and $\mathbb{R}$-structures on triangulated categories

There are the following structures on a triangulated category $\mathcal{D}$.

- A (bounded) t-structure $\mathcal{T}$ (torsion part of some torsion pair on $\mathcal{D}$ satisfying certain condition) or a heart $\mathcal{H}$, which provides a homology of $\mathcal{D}$.

$$\mathcal{D} = \langle \mathcal{H}[k] \mid k \in \mathbb{Z} \rangle.$$

- A refinement of t-structure is a slicing $\mathcal{P} = \{\mathcal{P}(\phi) \mid \phi \in \mathbb{R}\}$. Note that for any ϕ, $\mathcal{P}(\phi, +\infty)$ is a t-structure with heart $\mathcal{H} = \mathcal{P}(\phi, \phi + 1]$.

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$\mathbb{C}$-structure on triangulated categories

Definition (Bridgeland)

Let $\mathcal{D}$ be a triangulated category. A stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ consists of a central charge $Z\colon K(\mathcal{D}) \to \mathbb{C}$ and a slicing $\mathcal{P}(\phi)$ s.t.:

(a) *if* $0 \neq E \in \mathcal{P}(\phi)$, *then* $Z(E) = m_E e^{\mathbf{i}\pi\phi}$, $m_E \in \mathbb{R}_{>0}$,

(b) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$,

(c) *if* $\phi_1 > \phi_2$, *then* $\mathrm{Hom}_{\mathcal{D}}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) = 0$,

(d) *Any E admits a HN-filtration with factors* $\{A_i \in \mathcal{P}(\phi_i) \mid 1 \leq i \leq l\}$ *for real numbers* $\phi_1 > \cdots > \phi_l$.

Spaces of stability conditions

Usually, we assume that the Grothendieck group $K(\mathcal{D})$ is free of finite rank, i.e. $K(\mathcal{D}) \cong \mathbb{Z}^{\oplus n}$ for some n.

Also, there is a technical condition, the support property.

Theorem (Bridgeland)

The projection map of taking central charges

$$\mathcal{Z}\colon \operatorname{Stab}\mathcal{D} \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C}), \quad (Z, \mathcal{P}) \mapsto Z$$

is a local homeomorphism of topological spaces. In particular, $\mathcal{Z}$ *induces a complex structure on* $\operatorname{Stab}\mathcal{D}$.

$$\dim_{\mathbb{C}} \operatorname{Stab}\mathcal{D} = n.$$

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Actions on Stab $\mathcal{D}$

On Stab $\mathcal{D}$, there are two group actions commuting each other.

- The first one is the natural $\mathbb{C}$ action

$$s \cdot (Z, \mathcal{P}) = (Z \cdot e^{-\mathbf{i}\pi s}, \mathcal{P}_{\mathrm{Re}(s)}),$$

where $\mathcal{P}_x(\phi) = \mathcal{P}(\phi + x)$.

- There is also a natural action on Stab $\mathcal{D}$ induced by Aut $\mathcal{D}$, namely:

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$\mathbb{X}$-categories $\mathcal{D}_{\mathbb{X}}$

Let $\mathcal{D}_{\mathbb{X}}$ be a triangulated category with a distinguish auto-equivalence $\mathbb{X}\colon \mathcal{D}_{\mathbb{X}} \to \mathcal{D}_{\mathbb{X}}$. Let $E[l\mathbb{X}] := \mathbb{X}^l(E)$. Set

$$R = \mathbb{Z}[q^{\pm 1}]$$

and define the R-action on $K(\mathcal{D}_{\mathbb{X}})$ by

$$q^n \cdot [E] := [E[n\mathbb{X}]].$$

Then $K(\mathcal{D}_{\mathbb{X}})$ has an R-module structure and assume:

- $\mathrm{K}(\mathcal{D}_{\mathbb{X}}) \cong R^{\oplus n}$.

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$\mathbb{X}$-stability conditions

Definition (Ikeda-Qiu)

An $\mathbb{X}$-stability condition (σ, s) consists of a stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}_{\mathbb{X}}$ and a complex number $s \in \mathbb{C}$ satisfying

$$\mathbb{X}(\sigma) = s \cdot \sigma.$$

For a fixed complex number $s \in \mathbb{C}$, consider the specialization

$$q_s : \mathbb{C}[q, q^{-1}] \to \mathbb{C}, \quad q \mapsto e^{\mathbf{i}\pi s}.$$

Denote by $\mathbb{C}_s$ the complex numbers with the R-module structure through q_s.

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An $\mathbb{X}$-stability condition (σ, s) consists of a stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}_{\mathbb{X}}$ and a complex number $s \in \mathbb{C}$ satisfying

$$\mathbb{X}(\sigma) = s \cdot \sigma.$$

For a fixed complex number $s \in \mathbb{C}$, consider the specialization

$$q_s : \mathbb{C}[q, q^{-1}] \to \mathbb{C}, \quad q \mapsto e^{\mathbf{i}\pi s}.$$

Denote by $\mathbb{C}_s$ the complex numbers with the R-module structure through q_s.

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(e) *the slicing satisfies $\mathcal{P}(\phi + \mathrm{Re}(s)) = \mathcal{P}(\phi)[\mathbb{X}]$ for all $\phi \in \mathbb{R}$,*

(f) *the central charge $Z\colon K(\mathcal{D}_{\mathbb{X}}) \to \mathbb{C}_s$ is R-linear;*

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The $\mathbb{X}$-spaces.

With a updated technical condition: $\mathbb{X}$-support property.

Theorem (Ikeda-Qiu)

The projection map of taking central charges

$$\mathcal{Z}_s\colon \mathbb{X}\mathrm{Stab}_s\,\mathcal{D}_{\mathbb{X}} \longrightarrow \mathrm{Hom}_R(K(\mathcal{D}_{\mathbb{X}}), \mathbb{C}_s), \quad (Z, \mathcal{P}) \mapsto Z$$

is a local homeomorphism of topological spaces. In particular, $\mathcal{Z}_s$ induces a complex structure on $\mathbb{X}\mathrm{Stab}_s\,\mathcal{D}_{\mathbb{X}}$.

$$\dim_{\mathbb{C}} \mathbb{X}\mathrm{Stab}_s\,\mathcal{D}_{\mathbb{X}} = n.$$

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Example:Calabi-Yau-$\mathbb{X}$ dga $\Gamma_{\mathbb{X}} Q := (\mathbf{k}\overline{Q}, d)$

Definition (Ginzburg and Keller)

Let $Q = (Q_0, Q_1)$ be a finite acyclic quiver. Define a $\mathbb{Z} \oplus \mathbb{Z}\mathbb{X}$-graded quiver $\overline{Q}$ with same vertices and arrows

- *an original arrow $a: i \to j \in Q_1$ (degree 0);*
- *an opposite arrow $a^*: j \to i$ for the original arrow $a: i \to j \in Q_1$ (degree $2 - \mathbb{X}$);*
- *a loop t_i for each vertex $i \in Q_0$ (degree $1 - \mathbb{X}$).*

Let $\mathbf{k}\overline{Q}$ be a $\mathbb{Z} \oplus \mathbb{Z}\mathbb{X}$-graded path algebra of $\overline{Q}$, and define a differential $\mathrm{d}: \mathbf{k}\overline{Q} \to \mathbf{k}\overline{Q}$ of degree 1 by

- $\mathrm{d}\, a = \mathrm{d}\, a^* = 0$ *for $a \in Q_1$;*
- $\mathrm{d}\, t_i = e_i \left(\sum_{a \in Q_1} (aa^* - a^* a) \right) e_i$;

where e_i is the idempotent at $i \in Q_0$

Calabi-Yau-$\mathbb{X}$ categories

Denote by $\mathcal{D}_\infty(Q) := \mathcal{D}^b(\mathbf{k}Q)$ the bounded derived category of $\mathbf{k}Q$ and

$$\mathcal{D}_{\mathbb{X}}(Q) := \mathcal{D}_{fd}(\Gamma_{\mathbb{X}}Q)$$

the finite-dimensional derived category of $\Gamma_{\mathbb{X}}Q$.

Theorem (Keller, Van den Bergh)

The Calabi-Yau-$\mathbb{X}$ completion $\Pi_{\mathbb{X}}(kQ)$ of the path algebra kQ is isomorphic to the Ginzburg Calabi-Yau-$\mathbb{X}$ algebra $\Gamma_{\mathbb{X}}Q$. In particular, $\mathcal{D}_{\mathbb{X}}(Q)$ is Calabi-Yau-$\mathbb{X}$.

Outline

N-reduction

Let N be an integer. The orbit category $\mathcal{D}_{\mathbb{X}} /[\mathbb{X} - N]$ is *N-reductive* if it behaves well.

We will write $\mathcal{D}_N = \mathcal{D}_{\mathbb{X}} /\!/[\mathbb{X} - N]$ when the triangulated category $\mathcal{D}_N$ is the canonical triangulated hull of $\mathcal{D}_{\mathbb{X}} /[\mathbb{X} - N]$.

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N-reduction on spaces

Theorem (Ikeda-Qiu)

If $\mathcal{D}_{\mathbb{X}}$ is an N-reductive, then there is a canonical injection of complex manifolds

$$\iota_N \colon \mathbb{X}\mathrm{Stab}_N(\mathcal{D}_{\mathbb{X}}) \to \mathrm{Stab}\,\mathcal{D}_N,$$

whose image of ι_N is open and closed.

Outline

The $\mathbb{X}$-hearts

Definition

An $\mathbb{X}$-heart $\mathcal{D}_\infty \subset \mathcal{D}_\mathbb{X}$ is a full **triangulated** *subcategory of $\mathcal{D}_\mathbb{X}$ satisfying the following conditions:*

(1) *if $k_1 > k_2$, then $\mathrm{Hom}_{\mathcal{D}_\mathbb{X}}(\mathcal{D}_\infty[k_1\mathbb{X}], \mathcal{D}_\infty[k_2\mathbb{X}]) = 0$,*

(2) *Any E admits a HN-filtration with factors $\{A_i \in \mathcal{D}_\infty[k_i\mathbb{X}] \mid 1 \le k \le l\}$ with integers $k_1 > \cdots > k_l$.*

$$\mathcal{D}_\mathbb{X} = \langle \mathcal{D}_\infty[k] \mid k \in \mathbb{Z} \rangle.$$

$$K(\mathcal{D}_\infty) \otimes_\mathbb{Z} R \cong K(\mathcal{D}_\mathbb{X}).$$

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The construction

Consider a triple $(\mathcal{D}_\infty, \widehat{\sigma}, s)$ consists of an $\mathbb{X}$-heart $\mathcal{D}_\infty$, a (Bridgeland) stability condition $\widehat{\sigma} = (\widehat{Z}, \widehat{\mathcal{P}})$ on $\mathcal{D}_\infty$ and a complex number s. We construct a pre-stability condition $\sigma_* = (Z, \mathcal{P}_*)$ as follows. First extend $\widehat{Z}$ to

$$Z_q := \widehat{Z} \otimes R \colon K(\mathcal{D}_\mathbb{X}) \to \mathbb{C}[q, q^{-1}]$$

and let

$$Z = q_s \circ Z_q \colon K(\mathcal{D}_\mathbb{X}) \to \mathbb{C}$$

that gives a central charge function on $\mathcal{D}_\mathbb{X}$. The slicing $\mathcal{P}_*$ is defined as

$$\mathcal{P}_*(\phi) = \langle \widehat{\mathcal{P}}[\mathbb{Z}\mathbb{X}] \rangle^s := \langle \widehat{\mathcal{P}}(\phi - k\,\mathrm{Re}(s))[k\mathbb{X}] \rangle. \tag{3}$$

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Global dimension function

Generalizing the global dimension of an algebra (or an Abelian category), we have the following.

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Given a slicing $\mathcal{P}$ on a triangulated category $\mathcal{D}$. Define the global dimension of $\mathcal{P}$ by

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Some facts

- Given a heart $\mathcal{H}$ in $\mathcal{D}$, let $\mathcal{P}_0$ be the associated slicing with $\mathcal{P}(\phi) = \mathcal{H}[\phi]$ for $\phi \in \mathbb{Z}$ and $\mathcal{P}(\phi) = \emptyset$ otherwise. Then we have

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The inducing theorem

Theorem

Let $\mathcal{D}_{\mathbb{X}}$ be a Calabi-Yau-$\mathbb{X}$ category. Given a stability condition $\widehat{\sigma} = (\widehat{Z}, \widehat{\mathcal{P}})$ on an $\mathbb{X}$-heart $\mathcal{D}_\infty$ of $\mathcal{D}_{\mathbb{X}}$, then the induced extension pre-stability condition $\sigma_* = (Z, \mathcal{P}_*)$ is a stability condition on $\mathcal{D}_{\mathbb{X}}$ if and only if

$$\operatorname{gldim} \widehat{\sigma} \leq \operatorname{Re}(s) - 1. \tag{4}$$

The q-stability conditions

Definition

An (open) q-stability condition on $\mathcal{D}_{\mathbb{X}}$ is a pair (σ, s) consisting of a stability condition σ on $\mathcal{D}_{\mathbb{X}}$ and a complex parameter s, satisfying

- *σ is induced from some triple $(\mathcal{D}_\infty, \widehat{\sigma}, s)$ as above with*

$$\operatorname{gldim} \widehat{\sigma} + 1 < \operatorname{Re}(s).$$

Denote by $\operatorname{QStab}_s \mathcal{D}_{\mathbb{X}}$ the set of all q-stability conditions with the parameter $s \in \mathbb{C}$ and by $\operatorname{QStab} \mathcal{D}_{\mathbb{X}}$ the union of all $\operatorname{QStab}_s \mathcal{D}_{\mathbb{X}}$.

Theorem (Ikeda-Qiu)

$\operatorname{QStab} \mathcal{D}_{\mathbb{X}}$ is a complex manifold with dimension $n+1$.

On range of gldim

Theorem (Qiu)

Let Q be an acyclic quiver.

- *If Q is of Dynkin type. Then the range of* gldim *on* $\mathbb{C}\setminus \operatorname{Stab}\mathcal{D}_\infty(Q)/\operatorname{Aut}$ *is* $[1-2/h,+\infty)$, *where the unique minimal value is given by Kajiura-Saito-Takahashi's solution of Toda's Gepner equation*

$$\tau(\sigma)=(-2/h)\cdot\sigma.$$

Here h is the Coexter number of Q.

- *Otherwise, the range is* $[1,+\infty)$.

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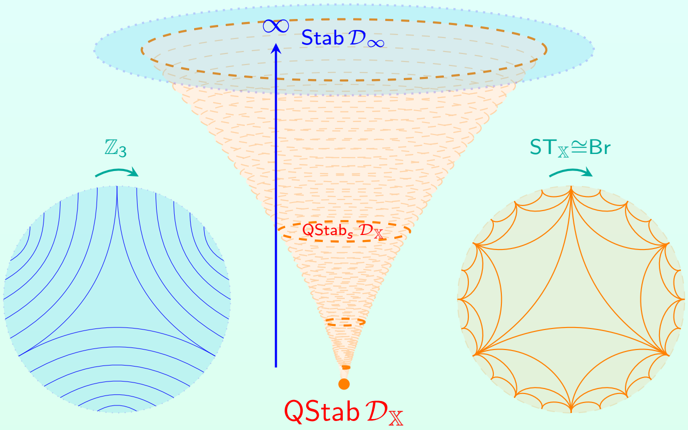
Case study: The tornado illustration for A_2



Outline

Quadratic differentials

Let $\mathbf{X}$ be a compact Riemann surface and $\omega_{\mathbf{X}}$ be its holomorphic cotangent bundle.

A *meromorphic quadratic differential* ϕ on $\mathbf{X}$ is a meromorphic section of the line bundle $\omega_{\mathbf{X}}^2$.

In terms of a local coordinate z on $\mathbf{X}$, such a ϕ can be written as $\phi(z) = g(z)\,dz^2$, where $g(z)$ is a meromorphic function.

At a point of $\mathbf{X}^\circ = \mathbf{X} \setminus \mathrm{Crit}(\phi)$, there is a distinguished local coordinate ω, uniquely defined up to transformations of the form $\omega \mapsto \pm\omega + \text{const}$, with respect to which $\phi(\omega) = d\omega \otimes d\omega$. In terms of a local coordinate z, we have $w = \int \sqrt{g(z)}dz$.

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Foliation and real blow-up

A (horizontal) *trajectory* of a quadratic differential ϕ on $\mathbf{X}^\circ$ is a maximal horizontal geodesic $\gamma\colon (0,1) \to \mathbf{X}^\circ$, with respect to the ϕ metric.

The trajectories of a meromeorphic quadratic differential ϕ provide the *horizontal foliation* on $\mathbf{X}$.

The real (oriented) blow-up of $(\mathbf{X}, \phi)$ is a differentiable surface $\mathbf{X}^\phi$, which is obtained from xx by replacing a pole $P \in \mathrm{Pol}(\phi)$ by a boundary ∂_P, consisting of the real tangent directions at P.

And we will mark the points on ∂_P that correspond to the distinguished tangent directions. Thus $\mathbf{X}^\phi$ is marked surface.

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Framed quadratic differentials (Calabi-Yau-3 case)

Definition

An $\mathbf{S}$*-framed quadratic differential* $(\mathbf{X}, \phi, \psi)$ *is a Riemann surface* $\mathbf{X}$ *with GMN differential* ϕ*, equipped with a diffeomorphism* $\psi\colon \mathbf{S} \to \mathbf{X}^{\phi}$*, preserving the marked points. Two* $\mathbf{S}$*-framed quadratic differentials* $(\mathbf{X}_i, \phi_i, \psi_i)$ *are equivalent, if there exists a biholomorphism* $f: \mathbf{X}_1 \to \mathbf{X}_2$ *s.t.*

- $f^*(\phi_2) = \phi_1$;
- $\psi_2^{-1} \circ f_* \circ \psi_1 \in \mathrm{Diff}_0(\mathbf{S})$, *where* $f_*\colon \mathbf{X}_1^{\phi_1} \to \mathbf{X}_2^{\phi_2}$ *is the induced diffeomorphism;*

GMN differential: all zeroes of ϕ are simple. (For simplicity, suppose that every pole of ϕ has order at least 3.)

Framed quadratic differentials (Calabi-Yau-3 case)

Definition

An $\mathbf{S}$*-framed quadratic differential* $(\mathbf{X}, \phi, \psi)$ *is a Riemann surface* $\mathbf{X}$ *with GMN differential* ϕ*, equipped with a diffeomorphism* $\psi\colon \mathbf{S} \to \mathbf{X}^{\phi}$*, preserving the marked points.*
Two $\mathbf{S}$*-framed quadratic differentials* $(\mathbf{X}_i, \phi_i, \psi_i)$ *are equivalent, if there exists a biholomorphism* $f : \mathbf{X}_1 \to \mathbf{X}_2$ *s.t.*

- $f^*(\phi_2) = \phi_1$;
- $\psi_2^{-1} \circ f_* \circ \psi_1 \in \mathrm{Diff}_0(\mathbf{S})$, *where* $f_*\colon \mathbf{X}_1^{\phi_1} \to \mathbf{X}_2^{\phi_2}$ *is the induced diffeomorphism;*

GMN differential: all zeroes of ϕ are simple. (For simplicity, suppose that every pole of ϕ has order at least 3.)

Decorated version

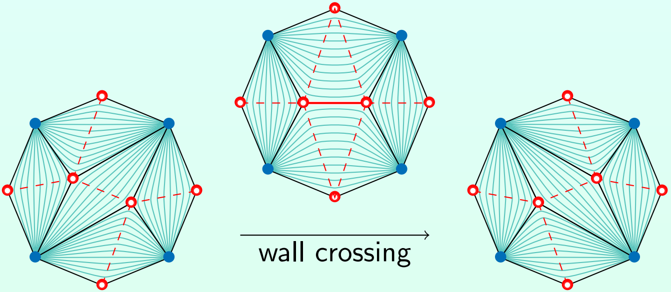
Definition (Qiu)

The decorated marked surface $\mathbf{S}_\triangle$ *is a marked surface* $\mathbf{S}$ *together with a fixed set* $\triangle$ *of* $\aleph$ *'decorating' points in the interior of* $\mathbf{S}$*, where* $\aleph$ *is the number of triangles in any triangulation of* $\mathbf{S}$*.*

Similarly, we have the $\mathbf{S}_\triangle$-framed version, where we require the decoration $\triangle$ maps to the set $\mathrm{Zero}(\phi)$ of the quadratic differential ϕ on $\mathbf{X}$.

WKB triangulation $\mathbf{T} = \mathbf{T}(\phi)$

Edges of $\mathbf{T}$ are open arcs/generic trajectories connecting marked points/poles.



Sketch of Bridgeland-Smith (similarly for HKK)

Given a quadratic differential ϕ on $\mathbf{S}_\triangle$ (i.e. in $\mathrm{FQuad}_3(\mathbf{S}_\triangle)$), we construct a stability condition σ in $\mathrm{Stab}^\circ \mathcal{D}_3(\mathbf{S})$ as follows:

- The WKB-triangulation determines a heart via cluster theory (FST + Keller-Nicolás).
- The closed arcs/saddle trajectories connecting decorating points/zeroes, correspond to the simples $\{S_i\}$ of hearts (cf. my previous series Decorated Marked Surfaces).
- The central charge is given by

$$Z(S_i) = \int_{\widetilde{\eta_i}} \sqrt{\phi},$$

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Outline

Yu Qiu

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Motivations
Mirror symmetry
Saito-Frobenius structure
Cluster theory

$\mathbb{X}$-stability conditions
Stability conditions
$\mathbb{X}$-stability conditions
N-reduction
q-stability conditions

Surface case
Quadratic differentials
Further studies

Upcoming preprints

Akishi Ikeda and I will construct quivers with superpotential from flat surfaces and q–quadratic differential and prove the following:

$$\mathrm{QQuad}_s^*(\log \mathbf{S}_\triangle) = \mathrm{QStab}_s^\circ \mathcal{D}_{\mathbb{X}}(\mathbf{S}_\triangle).$$

Together with Yu Zhou, we will generalize some of results of previous series of works on decorated marked surfaces to Calabi-Yau-$\mathbb{X}$ case.
See more in their talks.

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- 1807.00469
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Welcome to discuss questions with me via email (yu.qiu@bath.edu) or Facebook or Wechat (id: Q-dexter).

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Ending

Thank you!